MATH 2040 Lecture 14 (27/10/2016)

§ Adjoint operator (textbook § 6.3)

Recall: ∃ 1-1 correspondence (Fix $\dim_{\mathbb{F}} V = n$)

$$\left\{ \begin{array}{l} \text{linear operators} \\ T : V \to V \end{array} \right\} \xleftrightarrow[\beta]{\text{Fix basis}} M_{n\times n}(\mathbb{F})$$

$$\text{v. space} \left\{ \begin{array}{ll} T \pm S & A \pm B \\ cT & cA \\ T \circ S & AB \end{array} \right.$$

$$\beta\text{. O.N.B.} \left\{ \begin{array}{ll} T^* & \xleftrightarrow{\langle , \rangle} \quad A^t \\ \uparrow \text{adjoint of } T & \end{array} \right.$$

Def^n: Given $T : V \to V$ linear operator on an inner product space $(V, \langle , \rangle)$,

a linear operator $T^* : V \to V$ is called <u>the</u> adjoint of T

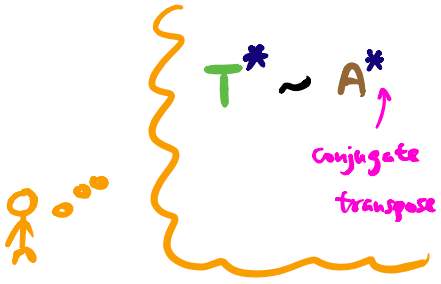
if $$\boxed{\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \forall\, x, y \in V.} \quad (*)$$

Thm: Assume V is finite dimensional.

Then, T^* <u>exists</u> and is <u>unique</u>!

Proof: Later.

Properties of adjoint:



(a) $(T \pm S)^* = T^* \pm S^*$

(b) $(cT)^* = \bar{c}\,T^*$

(c) $(T \cdot S)^* = S^* \cdot T^*$

(d) $T^{**} = T$

(e) $I^* = I$

Proof: (c). $\langle (T \cdot S)x, y \rangle \quad (= \langle x, (T \cdot S)^* y \rangle)$

$= \langle T(Sx), y \rangle$

$= \langle Sx, T^* y \rangle$

$= \langle x, S^*(T^* y) \rangle$

compare.

$\Downarrow$

$S^* \cdot T^* = (T \cdot S)^*$

Remark: Thm about existence / uniqueness "fails" in ∞-dim'l.

Theorem: β is an O.N.B for $(V, \langle , \rangle)$, $\dim V < +\infty$.

$\Rightarrow \quad [T]_\beta^* = [T^*]_\beta$

O.N.B.

E.g.: $T: \mathbb{R}^2 \to \mathbb{R}^2$ w/ std inner product.

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_2 \end{pmatrix}$$

Q: What is T^* ?

Fix $\beta = \{\vec{e}_1, \vec{e}_2\}$ std basis of $\mathbb{R}^2$ (O.N.B.)

$$[T]_\beta = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

What is $[T^*]_\beta$?

$$T^*(\vec{e}_1) = \langle T^*\vec{e}_1, \vec{e}_1 \rangle \vec{e}_1 + \langle T^*\vec{e}_1, \vec{e}_2 \rangle \vec{e}_2$$

$$T^*(\vec{e}_2) = \langle T^*\vec{e}_2, \vec{e}_1 \rangle \vec{e}_1 + \langle T^*\vec{e}_2, \vec{e}_2 \rangle \vec{e}_2$$

e.g. $\langle T^*\vec{e}_1, \vec{e}_1 \rangle \overset{(*)}{=} \langle \vec{e}_1, T\vec{e}_1 \rangle = 1$

repeating $\Rightarrow$ $[T^*]_\beta = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = [T]_\beta^t$

■

Ex: Show that it doesn't hold for β NOT O.N.B.

Indeed, the example above gives us a proof for the general case as well.

Why need O.N.B. β ?

$$[T]_\beta^t = [T^*]_\beta \qquad \beta = \text{std O.N.B of } \mathbb{R}^2$$

If I choose a different basis β'

$$[T]_{\beta'}^t = \left(Q^{-1}[T]_\beta Q\right)^t \qquad Q = \text{change of basis matrix.}$$

$$= Q^t [T]_\beta^t (Q^{-1})^t$$

$$= Q^t [T^*]_\beta (Q^{-1})^t$$

$$\stackrel{?}{=} [T^*]_{\beta'} = Q^{-1}[T^*]_\beta Q$$

"=" holds $\Leftrightarrow$ $Q^t = Q^{-1}$ $\Leftrightarrow$ $QQ^t = I$

Ex: β' O.N.B.